The average value inequality in sequential effect algebras*

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Abstract

A sequential effect algebra $(E, 0, 1, \oplus, \circ)$ is an effect algebra on which a sequential product $\circ$ with certain physics properties is defined, in particular, sequential effect algebra is an important model for studying quantum measurement theory. In 2005, Gudder asked the following problem: If $a, b \in (E, 0, 1, \oplus, \circ)$ and $a \perp b$ and $a \circ b \perp a \circ b$, is it the case that $2(a \circ b) \leq a^2 \oplus b^2$? In this paper, we construct an example to answer the problem negatively.

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Effect algebra was introduced in 1994 to model the quantum logic which may be fuzzy or unsharp, to be precise, an effect algebra is a system $(E, 0, 1, \oplus)$, where 0 and 1 are distinct elements of E and $\oplus$ is a partial binary operation on E satisfying [1]:

(EA1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.

(EA2) If $a \oplus (b \oplus c)$ is defined, then $(a \oplus b) \oplus c$ is defined and

$$(a \oplus b) \oplus c = a \oplus (b \oplus c).$$

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(EA3) For each $a \in E$, there exists a unique element $b \in E$ such that $a \oplus b = 1$.

(EA4) If $a \oplus 1$ is defined, then $a = 0$.

In an effect algebra $(E, 0, 1, \oplus)$, if $a \oplus b$ is defined, we write $a \perp b$. If $a \perp a$, we denote $a \oplus a$ by $2a$. For each $a \in (E, 0, 1, \oplus)$, it follows from (EA3) that there exists a unique element $b \in E$ such that $a \oplus b = 1$, we denote b by a'. If $a \wedge a' = 0$, we say that a is a *sharp element* of $(E, 0, 1, \oplus)$ (see [2]). Let $a, b \in (E, 0, 1, \oplus)$, if there exists a $c \in E$ such that $a \perp c$ and $a \oplus c = b$, then we say that $a \leq b$. It follows from [1] that $\leq$ is a partial order of $(E, 0, 1, \oplus)$ and satisfies that for each $a \in E$, $0 \leq a \leq 1$, $a \perp b$ iff $a \leq b'$.

In 2001, in order to study quantum measurement theory, Gudder began to consider the sequential product of two measurements A and B (see [3]). In 2002, Professors Gudder and Greechie introduced the abstract sequential effect algebra structure, that is:

A *sequential effect algebra* is an effect algebra $(E, 0, 1, \oplus)$ and another binary operation $\circ$ defined on $(E, 0, 1, \oplus)$ satisfying [4]:

(SEA1) The map $b \mapsto a \circ b$ is additive for each $a \in E$, that is, if $b \perp c$, then $a \circ b \perp a \circ c$ and $a \circ (b \oplus c) = a \circ b \oplus a \circ c$.

(SEA2) $1 \circ a = a$ for each $a \in E$.

(SEA3) If $a \circ b = 0$, then $a \circ b = b \circ a$.

(SEA4) If $a \circ b = b \circ a$, then $a \circ b' = b' \circ a$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for each $c \in E$.

(SEA5) If $c \circ a = a \circ c$ and $c \circ b = b \circ c$, then $c \circ (a \circ b) = (a \circ b) \circ c$ and $c \circ (a \oplus b) = (a \oplus b) \circ c$ whenever $a \perp b$.

Let $(E, 0, 1, \oplus, \circ)$ be a sequential effect algebra. Then the operation $\circ$ is said to be a sequential product on $(E, 0, 1, \oplus, \circ)$. If $a, b \in (E, 0, 1, \oplus, \circ)$ and $a \circ b = b \circ a$, then we say that a and b is *sequentially independent* and denoted by $a|b$ (see [4]). If $a \in (E, 0, 1, \oplus, \circ)$, we denote $a \circ a$ by a^2, it follows from ([4, Lemma 3.2]) that a is a sharp element of $(E, 0, 1, \oplus, \circ)$ iff $a^2 = a$. We denote the set of all sharp elements in $(E, 0, 1, \oplus, \circ)$ by E_s.

In 2005, in order to motivate the study of sequential effect algebra theory, Professor Gudder presented 25 important and interesting problems, the 23th problem asked ([5]): If $a, b \in (E, 0, 1, \oplus, \circ)$ and $a \perp b$ and $a \circ b \perp a \circ b$, is it the case that $2(a \circ b) \leq a^2 \oplus b^2$? In this paper, we construct an example to answer the problem negatively.

At first, we show that the above average value inequality does hold in the underlying sequential effect algebras under some additional conditions. That is:

Proposition 1. If $(E, 0, 1, \oplus, \circ)$ is a sequential effect algebra, $a, b \in E$, $a^2 \perp b^2$(a sufficient condition for this is $a \perp b$), $a \leq b$(or $b \leq a$) and $a|b$, then $(a \circ b) \perp (a \circ b)$ and $2(a \circ b) \leq a^2 \oplus b^2$.

Proof. Since $a \leq b$, there exists a $c \in E$ such that $a \oplus c = b$. Since $a|b$, it follows that $c|b$ (see [4] Lemma 3.1(v)).

$c \circ b = c \circ (a \oplus c) = (c \circ a) \oplus c^2$.

$b^2 = b \circ (a \oplus c) = (b \circ a) \oplus (b \circ c) = (a \circ b) \oplus (c \circ b) = (a \circ b) \oplus (c \circ a) \oplus c^2$.

Since $a^2 \perp b^2$, $a^2 \oplus b^2 = a^2 \oplus (a \circ b) \oplus (c \circ a) \oplus c^2$.

While $a \circ b = a \circ (a \oplus c) = a^2 \oplus (a \circ c) = a^2 \oplus (c \circ a)$, so $a^2 \oplus b^2 = (a \circ b) \oplus (a \circ b) \oplus c^2$.

It follows that $(a \circ b) \perp (a \circ b)$ and $2(a \circ b) \leq a^2 \oplus b^2$.

Finally, if $a \perp b$, it follows from $a^2 \leq a$ and $b^2 \leq b$ that $a^2 \perp b^2$. The proposition is proved.

Proposition 2. If $(E, 0, 1, \oplus, \circ)$ is a sequential effect algebra, $a, b \in E$, $a \perp b$, $a \in E_s$(or $b \in E_s$), then $(a \circ b) \perp (a \circ b)$ and $2(a \circ b) \leq a^2 \oplus b^2$.

Proof. Since $a \perp b$ and $a \in E_s$, it follows that $a \circ b = 0$ (see [4] Lemma 3.3(ii)), so $2(a \circ b) \leq a^2 \oplus b^2$.

Now, we construct a sequential effect algebra to show that the above average value inequality does not always hold.

In this paper, we denote $\mathbf{Z}$ the integer set, $\mathbf{N}$ the nonnegative integer set and $\mathbf{N}^+$ the positive integer set.

Let $E_0 = \{0, 1, a_n, b_n, c_{i,k,m}, d_{i,k,m} |\ n \in \mathbf{N}^+, i, k \in \mathbf{N} \textit{ and } i^2 + k^2 \neq 0, m \in \mathbf{Z}\}$.

For simplicity, in the sequel, unless specified, the subindex of respective elements will always take values in the corresponding default sets. To be accurately, when we write a_n, b_n, n always take values in $\mathbf{N}^+$, when we write $c_{i,k,m}, d_{i,k,m}$, i, k always take values in $\mathbf{N}$ and $i^2 + k^2 \neq 0$ and m always take values in $\mathbf{Z}$.

We define a partial binary operation $\oplus$ on E_0 as follows(when we write $x \oplus y = z$, we always mean $x \oplus y = z = y \oplus x$):

For each $x \in E_0$, $0 \oplus x = x$,

$a_n \oplus a_m = a_{n+m}$, $a_n \oplus c_{i,k,m} = c_{i,k,n+m}$, $a_n \oplus d_{i,k,m} = d_{i,k,m-n}$, $c_{i,k,m} \oplus c_{r,s,t} = c_{i+r,k+s,m+t}$.

For $n < m$, $a_n \oplus b_m = b_{m-n}$, $a_n \oplus b_n = 1$.

For $i \leq r$ *and* $k \leq s$ *and* $(r-i)^2 + (s-k)^2 \neq 0$. $c_{i,k,m} \oplus d_{r,s,t} = d_{r-i,s-k,t-m}$.

For $i = r$ *and* $k = s$ *and* $m < t$, $c_{i,k,m} \oplus d_{r,s,t} = b_{t-m}$.

For $i = r$ *and* $k = s$ *and* $m = t$, $c_{i,k,m} \oplus d_{r,s,t} = 1$.

No other $\oplus$ operation is defined.

Next, we define a binary operation $\circ$ on E_0 as follows(when we write $x \circ y = z$, we always mean $x \circ y = z = y \circ x$):

For each $x \in E_0$, $0 \circ x = 0$, $1 \circ x = x$,

$a_n \circ a_m = 0$, $a_n \circ b_m = a_n$, $b_n \circ b_m = b_{m+n}$, $a_n \circ c_{i,k,m} = 0$, $c_{i,k,m} \circ b_n = c_{i,k,m}$, $a_n \circ d_{i,k,m} = a_n$, $b_n \circ d_{i,k,m} = d_{i,k,m+n}$, $d_{i,k,m} \circ d_{r,s,t} = d_{i+r,k+s,m+t-is-kr}$, $c_{i,k,m} \circ d_{r,s,t} = c_{i,k,m-is-kr}$, $c_{i,k,m} \circ c_{r,s,t} = a_{is+kr}(when\ is + kr \neq 0)$ *or* $0(when\ is + kr = 0)$.

Proposition 3. $(E_0, 0, 1, \oplus, \circ)$ is a sequential effect algebra.

Proof. First we verify that $(E_0, 0, 1, \oplus)$ is an effect algebra.

(EA1) and (EA4) are trivial.

We verify (EA2), we omit the trivial cases about 0,1:

$a_n \oplus (a_m \oplus a_k) = (a_n \oplus a_m) \oplus a_k = a_{k+m+n}$.

$a_n \oplus (a_m \oplus c_{i,j,k}) = (a_n \oplus a_m) \oplus c_{i,j,k} = c_{i,j,k+m+n}$.

$a_n \oplus (a_m \oplus d_{i,j,k}) = (a_n \oplus a_m) \oplus d_{i,j,k} = d_{i,j,k-m-n}$.

$a_n \oplus (c_{r,s,t} \oplus c_{i,j,k}) = (a_n \oplus c_{r,s,t}) \oplus c_{i,j,k} = c_{i+r,s+j,k+t+n}$.

$c_{l,m,n} \oplus (c_{r,s,t} \oplus c_{i,j,k}) = (c_{l,m,n} \oplus c_{i,j,k}) \oplus c_{r,s,t} = c_{i+l+r,j+m+s,k+n+t}$.

Each $a_n \oplus (a_m \oplus b_k)$ or $(a_n \oplus a_m) \oplus b_k$ is defined iff $n + m \leq k$, at this case,

$a_n \oplus (a_m \oplus b_k) = (a_n \oplus a_m) \oplus b_k = b_{k-m-n}(when\ m+n < k)\ or\ 1(when\ m+n = k)$.

Each $a_n \oplus (c_{r,s,t} \oplus d_{i,j,k})$ or $(a_n \oplus c_{r,s,t}) \oplus d_{i,j,k}$ or $(a_n \oplus d_{i,j,k}) \oplus c_{r,s,t}$ is defined iff one of the following two conditions is satisfied:

(1) $r \leq i\ and\ s \leq j\ and\ (i-r)^2 + (j-s)^2 \neq 0$, at this case, $a_n \oplus (c_{r,s,t} \oplus d_{i,j,k}) = (a_n \oplus c_{r,s,t}) \oplus d_{i,j,k} = (a_n \oplus d_{i,j,k}) \oplus c_{r,s,t} = d_{i-r,j-s,k-t-n}$;

(2) $r = i\ and\ s = j\ and\ n+t \leq k$, at this case, $a_n \oplus (c_{r,s,t} \oplus d_{i,j,k}) = (a_n \oplus c_{r,s,t}) \oplus d_{i,j,k} = (a_n \oplus d_{i,j,k}) \oplus c_{r,s,t} = b_{k-t-n}(when\ n+t < k)\ or\ 1(when\ n+t = k)$.

Each $c_{l,m,n} \oplus (c_{r,s,t} \oplus d_{i,j,k})$ or $(c_{l,m,n} \oplus c_{r,s,t}) \oplus d_{i,j,k}$ is defined iff one of the following two conditions is satisfied:

(1) $l+r \leq i\ and\ m+s \leq j\ and\ (i-l-r)^2 + (j-m-s)^2 \neq 0$, at this case, $c_{l,m,n} \oplus (c_{r,s,t} \oplus d_{i,j,k}) = (c_{l,m,n} \oplus c_{r,s,t}) \oplus d_{i,j,k} = d_{i-l-r,j-m-s,k-t-n}$;

(2) $l+r = i\ and\ m+s = j\ and\ n+t \leq k$, at this case, $c_{l,m,n} \oplus (c_{r,s,t} \oplus d_{i,j,k}) = (c_{l,m,n} \oplus c_{r,s,t}) \oplus d_{i,j,k} = b_{k-t-n}(when\ n+t < k)\ or\ 1(when\ n+t = k)$.

We verify (EA3): $a_n \oplus b_n = 1$, $c_{i,k,m} \oplus d_{i,k,m} = 1$.

So $(E_0, 0, 1, \oplus)$ is an effect algebra.

We now verify that $(E_0, 0, 1, \oplus, \circ)$ is a sequential effect algebra.

(SEA2) and (SEA3) and (SEA5) are trivial.

We verify (SEA1), we omit the trivial cases about 0,1:

$a_n \circ (a_m \oplus a_k) = a_n \circ a_m \oplus a_n \circ a_k = 0$,

$b_n \circ (a_m \oplus a_k) = b_n \circ a_m \oplus b_n \circ a_k = a_{m+k}$,

$c_{r,s,t} \circ (a_m \oplus a_k) = c_{r,s,t} \circ a_m \oplus c_{r,s,t} \circ a_k = 0$,

$d_{r,s,t} \circ (a_m \oplus a_k) = d_{r,s,t} \circ a_m \oplus d_{r,s,t} \circ a_k = a_{m+k}$.

$a_n \circ (a_m \oplus c_{r,s,t}) = a_n \circ a_m \oplus a_n \circ c_{r,s,t} = 0$,

$b_n \circ (a_m \oplus c_{r,s,t}) = b_n \circ a_m \oplus b_n \circ c_{r,s,t} = c_{r,s,m+t}$,

$c_{x,y,z} \circ (a_m \oplus c_{r,s,t}) = c_{x,y,z} \circ a_m \oplus c_{x,y,z} \circ c_{r,s,t} = a_{xs+yr}(when\ xs+yr \neq 0)\ or\ 0(when\ xs+yr = 0)$,

$d_{x,y,z} \circ (a_m \oplus c_{r,s,t}) = d_{x,y,z} \circ a_m \oplus d_{x,y,z} \circ c_{r,s,t} = c_{r,s,m+t-xs-yr}$.

$a_n \circ (a_m \oplus d_{r,s,t}) = a_n \circ a_m \oplus a_n \circ d_{r,s,t} = a_n$,

$b_n \circ (a_m \oplus d_{r,s,t}) = b_n \circ a_m \oplus b_n \circ d_{r,s,t} = d_{r,s,n+t-m}$,

$c_{x,y,z} \circ (a_m \oplus d_{r,s,t}) = c_{x,y,z} \circ a_m \oplus c_{x,y,z} \circ d_{r,s,t} = c_{x,y,z-xs-yr}$,

$d_{x,y,z} \circ (a_m \oplus d_{r,s,t}) = d_{x,y,z} \circ a_m \oplus d_{x,y,z} \circ d_{r,s,t} = d_{x+r,y+s,z+t-m-xs-yr}$.

$a_n \circ (c_{x,y,z} \oplus c_{r,s,t}) = a_n \circ c_{x,y,z} \oplus a_n \circ c_{r,s,t} = 0$,

$b_n \circ (c_{x,y,z} \oplus c_{r,s,t}) = b_n \circ c_{x,y,z} \oplus b_n \circ c_{r,s,t} = c_{x+r,y+s,z+t}$,

$c_{i,k,m} \circ (c_{x,y,z} \oplus c_{r,s,t}) = c_{i,k,m} \circ c_{x,y,z} \oplus c_{i,k,m} \circ c_{r,s,t} = a_{i(y+s)+k(x+r)} (when\ i(y+s)+k(x+r) \neq 0)\ or\ 0 (when\ i(y+s)+k(x+r) = 0)$,

$d_{i,k,m} \circ (c_{x,y,z} \oplus c_{r,s,t}) = d_{i,k,m} \circ c_{x,y,z} \oplus d_{i,k,m} \circ c_{r,s,t} = c_{x+r,y+s,z+t-i(y+s)-k(x+r)}$.

For $m \leq k$,

$a_n \circ (a_m \oplus b_k) = a_n \circ a_m \oplus a_n \circ b_k = a_n$,

$b_n \circ (a_m \oplus b_k) = b_n \circ a_m \oplus b_n \circ b_k = b_{n+k-m}$,

$c_{x,y,z} \circ (a_m \oplus b_k) = c_{x,y,z} \circ a_m \oplus c_{x,y,z} \circ b_k = c_{x,y,z}$,

$d_{x,y,z} \circ (a_m \oplus b_k) = d_{x,y,z} \circ a_m \oplus d_{x,y,z} \circ b_k = d_{x,y,z+k-m}$.

For $i \leq r\ and\ k \leq s\ and\ (r-i)^2 + (s-k)^2 \neq 0$,

$a_n \circ (c_{i,k,m} \oplus d_{r,s,t}) = a_n \circ c_{i,k,m} \oplus a_n \circ d_{r,s,t} = a_n$,

$b_n \circ (c_{i,k,m} \oplus d_{r,s,t}) = b_n \circ c_{i,k,m} \oplus b_n \circ d_{r,s,t} = d_{r-i,s-k,n+t-m}$,

$c_{x,y,z} \circ (c_{i,k,m} \oplus d_{r,s,t}) = c_{x,y,z} \circ c_{i,k,m} \oplus c_{x,y,z} \circ d_{r,s,t} = c_{x,y,z-x(s-k)-y(r-i)}$,

$d_{x,y,z} \circ (c_{i,k,m} \oplus d_{r,s,t}) = d_{x,y,z} \circ c_{i,k,m} \oplus d_{x,y,z} \circ d_{r,s,t} = d_{x+r-i,y+s-k,z+t-m-x(s-k)-y(r-i)}$.

For $i = r\ and\ k = s\ and\ m \leq t$,

$a_n \circ (c_{i,k,m} \oplus d_{r,s,t}) = a_n \circ c_{i,k,m} \oplus a_n \circ d_{r,s,t} = a_n$,

$b_n \circ (c_{i,k,m} \oplus d_{r,s,t}) = b_n \circ c_{i,k,m} \oplus b_n \circ d_{r,s,t} = b_{n+t-m}$,

$c_{x,y,z} \circ (c_{i,k,m} \oplus d_{r,s,t}) = c_{x,y,z} \circ c_{i,k,m} \oplus c_{x,y,z} \circ d_{r,s,t} = c_{x,y,z}$,

$d_{x,y,z} \circ (c_{i,k,m} \oplus d_{r,s,t}) = d_{x,y,z} \circ c_{i,k,m} \oplus d_{x,y,z} \circ d_{r,s,t} = d_{x,y,z+t-m}$.

We verify (SEA4), we omit the trivial cases about 0,1:

$a_n \circ (a_m \circ a_k) = (a_n \circ a_m) \circ a_k = 0$,

$a_n \circ (a_m \circ b_k) = b_k \circ (a_n \circ a_m) = a_m \circ (a_n \circ b_k) = 0$,

$a_n \circ (a_m \circ c_{r,s,t}) = c_{r,s,t} \circ (a_n \circ a_m) = a_m \circ (a_n \circ c_{r,s,t}) = 0$,

$a_n \circ (a_m \circ d_{r,s,t}) = d_{r,s,t} \circ (a_n \circ a_m) = a_m \circ (a_n \circ d_{r,s,t}) = 0$,

$a_n \circ (b_m \circ b_k) = b_k \circ (a_n \circ b_m) = b_m \circ (a_n \circ b_k) = a_n$,

$a_n \circ (b_m \circ c_{r,s,t}) = c_{r,s,t} \circ (a_n \circ b_m) = b_m \circ (a_n \circ c_{r,s,t}) = 0$,

$a_n \circ (b_m \circ d_{r,s,t}) = d_{r,s,t} \circ (a_n \circ b_m) = b_m \circ (a_n \circ d_{r,s,t}) = a_n$,

$a_n \circ (c_{i,k,m} \circ c_{r,s,t}) = c_{r,s,t} \circ (a_n \circ c_{i,k,m}) = c_{i,k,m} \circ (a_n \circ c_{r,s,t}) = 0$,

$a_n \circ (c_{i,k,m} \circ d_{r,s,t}) = d_{r,s,t} \circ (a_n \circ c_{i,k,m}) = c_{i,k,m} \circ (a_n \circ d_{r,s,t}) = 0,$

$a_n \circ (d_{i,k,m} \circ d_{r,s,t}) = d_{r,s,t} \circ (a_n \circ d_{i,k,m}) = d_{i,k,m} \circ (a_n \circ d_{r,s,t}) = a_n,$

$b_n \circ (b_m \circ b_k) = b_k \circ (b_n \circ b_m) = b_{m+n+k},$

$b_n \circ (b_m \circ c_{r,s,t}) = c_{r,s,t} \circ (b_n \circ b_m) = b_m \circ (b_n \circ c_{r,s,t}) = c_{r,s,t},$

$b_n \circ (b_m \circ d_{r,s,t}) = d_{r,s,t} \circ (b_n \circ b_m) = b_m \circ (b_n \circ d_{r,s,t}) = d_{r,s,n+m+t},$

$b_n \circ (c_{i,k,m} \circ c_{r,s,t}) = c_{r,s,t} \circ (b_n \circ c_{i,k,m}) = c_{i,k,m} \circ (b_n \circ c_{r,s,t}) = a_{is+kr}(when\ is+kr \neq 0)\ or\ 0(when\ is+kr=0),$

$b_n \circ (c_{i,k,m} \circ d_{r,s,t}) = d_{r,s,t} \circ (b_n \circ c_{i,k,m}) = c_{i,k,m} \circ (b_n \circ d_{r,s,t}) = c_{i,k,m-is-kr},$

$b_n \circ (d_{i,k,m} \circ d_{r,s,t}) = d_{r,s,t} \circ (b_n \circ d_{i,k,m}) = d_{i,k,m} \circ (b_n \circ d_{r,s,t}) = d_{i+r,k+s,n+m-t-is-kr},$

$c_{x,y,z} \circ (c_{i,k,m} \circ c_{r,s,t}) = c_{r,s,t} \circ (c_{x,y,z} \circ c_{i,k,m}) = 0,$

$c_{x,y,z} \circ (c_{i,k,m} \circ d_{r,s,t}) = d_{r,s,t} \circ (c_{x,y,z} \circ c_{i,k,m}) = c_{i,k,m} \circ (c_{x,y,z} \circ d_{r,s,t}) = a_{xk+yi}(when\ xk+yi \neq 0)\ or\ 0(when\ xk+yi=0),$

$c_{x,y,z} \circ (d_{i,k,m} \circ d_{r,s,t}) = d_{r,s,t} \circ (c_{x,y,z} \circ d_{i,k,m}) = d_{i,k,m} \circ (c_{x,y,z} \circ d_{r,s,t}) = c_{x,y,z-x(k+s)-y(i+r)},$

$d_{x,y,z} \circ (d_{i,k,m} \circ d_{r,s,t}) = d_{r,s,t} \circ (d_{x,y,z} \circ d_{i,k,m}) = d_{x+i+r,y+k+s,z+m+t-(is+kr+xk+xs+yi+yr)}.$

So $(E_0, 0, 1, \oplus, \circ)$ is a sequential effect algebra.

Our main result is:

Theorem 1. The average value inequality does not always hold in sequential effect algebras.

Proof. In fact, in $(E_0, 0, 1, \oplus, \circ)$, $c_{1,0,0} \perp c_{0,1,0}$, $c_{1,0,0} \oplus c_{0,1,0} = c_{1,1,0}$. $c_{1,0,0} \circ c_{0,1,0} = a_1$, $a_1 \perp a_1$, $a_1 \oplus a_1 = a_2$. But $2(c_{1,0,0} \circ c_{0,1,0}) = 2a_1 = a_1 \oplus a_1 = a_2$, $(c_{1,0,0})^2 = c_{1,0,0} \circ c_{1,0,0} = 0$, $(c_{0,1,0})^2 = c_{0,1,0} \circ c_{0,1,0} = 0$, so $2(c_{1,0,0} \circ c_{0,1,0}) \not\leq (c_{1,0,0})^2 \oplus (c_{0,1,0})^2$.

Remarks. Recently, the 2th problem, the 3th problem, the 17th problem and the 20th problem of Gudder have also been answered ([6-9]).

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